

THE NORMAL COMPONENT OF THE INDUCED VELOCITY NEAR  
A VORTEX RING AND ITS APPLICATION  
TO LIFTING ROTOR PROBLEMS

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Jacob Henri de Leeuw

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# NOMENCLATURE

$x\ y\ z$	coordinate system.	$\psi$	stream function.
$h\ k\ \ell$	values of $x$ , $y$ and $z$ -- coordinates of a point, $P$ .	$\phi$	potential.
$v$	induced velocity.	$\omega$	solid angle.
$R$	radius of vortex ring, radius of rotor.	$\Gamma$	circulation strength.
$r_2$	greatest distance from point, $P$ , to vortex ring.	$\chi$	wake angle.
$r_1$	least distance from point, $P$ , to vortex ring	$\Delta_1$	contribution by finite part of wake.
		$\Delta_2$	contribution by remaining part of wake.
		$\lambda$	parameter in elliptic integrals.

## Subscripts

$x\ y$	component in $x$ , $y$ direction.	$p$	value at the point, $P$ .
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SUMMARY

Expressions are presented for the velocity components in the flow field of a vortex ring. Values of the normal component of the velocity were calculated and are presented in tabular form.

A method, employing the calculated tabular values, is presented for computing the approximate values of the normal component of the induced velocity in the vicinity of a helicopter rotor.

## CHAPTER I

### INTRODUCTION

In order to calculate the forces on a helicopter rotor, it is essential to know the magnitude of the induced velocities, especially of the component normal to the rotor, at all positions on the rotor disk. These velocities may be induced by the bound and free vortices of the rotor itself, in the case of a single rotor helicopter, or, in the case of a helicopter with two or more rotors, there will also be an influence on the induced velocity in each rotor plane due to the vortex field of the others.

An actual rotor has a finite number of blades, each of which has, in general, a nonuniform but continuous distribution of circulation along its span. Thus vortices of infinitesimal strength are shed from each point of the blade trailing edges. Because of the change in velocity of the blade relative to the air during each revolution of the rotor, the conditions of equilibrium require the blade circulation to change periodically. This results in a cyclic change of the strength and distribution of vorticity shed from the blade. Also, the velocities with which the vortices leave the trailing edge of the blade vary in magnitude and direction during any revolution. Furthermore, in the general case, the wake may not have a straight axis or a constant cross section.

The foregoing description applies to the usual steady-state flight



conditions. There are special cases, however, where additional complications occur.

Due to the involved character of the rotor wake, the exact calculation of the induced velocity is exceedingly difficult, and at present mathematical analysis can only be usefully applied after introducing a very simplified wake picture. Coleman, Feingold and Stempin (1) made the assumption that the wake could be considered as a straight, elliptic cylinder, consisting of a continuous distribution of ring vortices lying in planes parallel to the rotor plane. This wake picture omits most of the complicating aspects of the actual wake, but it effectively leads to a mathematical expression for the induced velocity at any point in the flow field. Unfortunately, this expression is not useful for direct practical application because it can be reduced in terms of known functions only for the special case of points on the longitudinal axis in the rotor disk.

This simplified wake suggests, however, that if the velocities in the field of a vortex ring were known, it would be possible to arrive at approximate values for the induced velocity at any arbitrary point in the vicinity of a rotor by numerical methods. The present paper derives the necessary expressions for the calculation of the velocity components in the flow field of a vortex ring. Tabular values are given for the normal component of the velocity in the ring field, since it is the normal component of the induced velocity which is of prime importance in the application to lifting rotor problems.

The use of the tabulated values is demonstrated for the case of an inclined wake where the induced velocity normal to the rotor plane is calculated at several points in the vicinity of the rotor disk.

## CHAPTER II

## THE CIRCULAR VORTEX

In the case of a single circular vortex, it may be seen that the flow is axially symmetric. The flow field of such a vortex is therefore completely described if the flow in one of the radial planes is known. This makes it possible to use as an orthogonol coordinate system the axis of the circular vortex as x-axis and any line perpendicular to the x-axis as the y-axis. It is convenient to let the origin of the coordinate system lie at the center of the vortex. Because of the axial symmetry of the flow field, there exists a stream function for this three-dimensional flow in a complete analogy with the usual two-dimensional cases. Let this stream function,  $\psi$ , be defined as by Lamb (2). Then the values of the velocity components at the point, P (x,y), are given by

$$v_x = -\frac{1}{y} \frac{\partial \psi}{\partial y} \quad (1)$$

$$v_y = \frac{1}{y} \frac{\partial \psi}{\partial x} \quad (2)$$

If the least and greatest distances from the point, P, to the vortex ring are denoted by  $r_1$  and  $r_2$  respectively, then the stream function at the point, P, is given by

$$\psi = -\frac{\Gamma}{2\pi} (r_1 + r_2) \{K(\lambda) - E(\lambda)\} \quad (3)$$

where

$$\lambda = \frac{r_2 - r_1}{r_2 + r_1}, \quad (4)$$

$K(\lambda)$  and  $E(\lambda)$  stand for the "complete" elliptic integrals of first and second kind respectively, and  $\Gamma$  is the circulation strength of the vortex ring (3).

According to equations (1) and (2), the velocity components can be found by differentiation of the expression for the stream function. In Appendix II these differentiations are performed and the results may be given as

$$v_x = \frac{\Gamma}{2\pi y} (AB + CDE) \quad (5)$$

$$v_y = -\frac{\Gamma}{2\pi y} (AB' + CDE') \quad (6)$$

where

$$A = K(\lambda) - E(\lambda)$$

$$B = \frac{y - R}{\sqrt{x^2 + (y - R)^2}} + \frac{y + R}{\sqrt{x^2 + (y + R)^2}}$$

$R = \text{radius of vortex ring}$

$$C = r_1 + r_2 = \sqrt{x^2 + (y - R)^2} + \sqrt{x^2 + (y + R)^2}$$

$$D = \frac{\lambda E(\lambda)}{1 - \lambda^2}$$

$$\begin{aligned}
E &= \frac{1}{R} - \frac{(x^2 + y^2 + R^2) - \sqrt{x^2 + (y - R)^2} \sqrt{x^2 + (y + R)^2}}{2Ry^2} \\
&\quad - \frac{(y + R) \{x^2 + (y - R)^2\} + (y - R) \{x^2 + (y + R)^2\}}{2Ry \sqrt{x^2 + (y - R)^2} \sqrt{x^2 + (y + R)^2}} \\
B' &= \frac{y}{\sqrt{x^2 + (y - R)^2}} + \frac{y}{\sqrt{x^2 + (y + R)^2}} \\
E' &= \frac{x}{Ry} \left\{ 1 - \frac{y^2 + x^2 + R^2}{\sqrt{x^2 + (y - R)^2} \sqrt{x^2 + (y + R)^2}} \right\}
\end{aligned}$$

Equations (5) and (6) yield the velocity components at all points, P, except at those with  $y = 0$ , in which case the expressions become indeterminate.

It is convenient to determine the velocities at those points by using a different approach. The potential at any point,  $P(x, y)$ , due to a closed vortex can be given as

$$\phi_p = \frac{\Gamma}{4\pi} \omega \quad (7)$$

where  $\omega$  is the solid angle at P subtended by the closed vortex (4). The solid angle may be defined as  $4\pi$  times the ratio of the volume cut from the unit sphere around P by the cone, determined by the point, P, and the closed vortex, to the volume of the unit sphere. In the case of points

on the axis of symmetry of the vortex ring, i.e., for points with  $y = 0$ , the solid angle may be expressed in the notation of figure 1 by

$$\omega = \frac{4\pi}{\frac{4}{3}\pi x l^3} \int_0^R \frac{\frac{1}{3}x^2 \pi r dr}{\sqrt{(x^2 + r^2)^3}}. \quad (8)$$

Equation (8) integrates to

$$\omega = 2\pi \left( 1 - \frac{x}{\sqrt{x^2 + R^2}} \right) \quad (9)$$

The value of  $\omega$  given by equation (9) when substituted in equation (7) yields

$$\phi_p = \frac{\Gamma}{2} \left( 1 - \frac{x}{\sqrt{x^2 + R^2}} \right) \quad (10)$$

Then from equation (1) it is finally found that

$$v_{xp} = -\frac{\Gamma}{2} \frac{R^2}{\sqrt{(x^2 + R^2)^3}} = -\frac{\Gamma}{2} \frac{R^2}{\sqrt{(x^2 + R^2)^3}}. \quad (11)$$

By use of equations (5) and (11) the velocity component,  $v_x$ , can now be found at every point in the field. Special mention may be made of the points  $(0, R)$ . Because the value of the elliptic integral of the first kind approaches infinity when its argument approaches one, the value of

the velocity in the plane of the ring will approach infinity as  $y$  approaches  $R$ .

Values of the velocity component normal to the plane of the ring were calculated for a set of points for a unit ring radius,  $R$ , and a vortex strength  $\Gamma = -1$ . The values of the elliptic integrals that occur in the formula for  $v_x$  were found by interpolation from a table which gave values of the integrals to ten decimal places for arguments with an increment of 0.01 (5). The procedure of the interpolation that was used is described by Peirce (6). The calculations were arranged in tabular form, where each table was for a fixed value of  $y$  and a suitably chosen set of  $x$ -values. Table 1 (Appendix I) gives the results of the calculations.

A check was made on the calculations by means of the following analysis. A continuous cylinder of vortex rings of equal radius,  $R = 1$ , was taken with the  $x$ -axis as the common axis of the vortex rings. This cylinder extended from  $x = 0$  to  $x = \infty$ . The potential at any point of the flow field that is induced by this infinite succession of vortex rings may be expressed as the integral of the potentials induced at that point by each individual vortex ring of infinitesimal strength,  $\frac{d\Gamma}{dx} dx$ . If the point,  $P$ , is given by the coordinates  $(h, k)$ , this integral may be expressed as

$$\phi_P = \int_0^{\infty} \frac{1}{4\pi} \frac{d\Gamma}{dx} \omega(x - h, k) dx \quad (12)$$

where  $\omega(x - h, k)$ , the solid angle at  $P$  subtended by a vortex ring at



distance  $x$  from the origin of the coordinate system, is clearly a function of the distances,  $(x - h)$  and  $k$ , from the center of the vortex ring to the point,  $P$ . Since all the ring centers have the same distances in the  $y$ -direction to the point,  $P$ , it is possible to express the integral with a new coordinate system where the  $y$ -axis goes through the point,  $P$ , and the origin is on the axis of the cylinder. This change will only alter the limits of the integral and it may be seen that equation (12) may then be written

$$\phi_p = \int_h^{\infty} \frac{1}{4\pi} \frac{d\Gamma}{dx} \omega(x, k) dx. \quad (13)$$

Since the velocity component at  $P$  is the derivative of the potential in the desired direction, at that point the component of the velocity in the  $x$ -direction is

$$v_{x_p} = \frac{\partial}{\partial h} \int_h^{\infty} \frac{1}{4\pi} \frac{d\Gamma}{dx} \omega(x, k) dx. \quad (14)$$

Then, according to a mathematical theorem (7), equation (14) reduces to

$$v_{x_p} = - \frac{1}{4\pi} \frac{d\Gamma}{dx} \omega(h, k). \quad (15)$$

For the case of  $h = 0$ , the preceding expression reduces to

$$v_{x_p} = - \frac{1}{4\pi} \frac{d\Gamma}{dx} \omega(0, k). \quad (16)$$

It is known that

$$\begin{aligned}\omega(0,k) &= 2\pi & k < \text{radius of the ring}, \\ \omega(0,k) &= 0 & k > \text{radius of the ring}.\end{aligned}\tag{17}$$

This then leads to the conclusion that in the case of  $\frac{d\Gamma}{dx} = -1$ , the velocity component  $v_x$ , in the plane  $h = 0$ , due to the infinite system of vortex rings will be

$$\begin{aligned}v_{x_p}(0,k) &= 1/2 & k < R, \\ v_{x_p}(0,k) &= 0 & k > R.\end{aligned}\tag{18}$$

The values of  $v_x$  for points in the plane  $x = 0$ , because of the cylindrical system of vortex rings, may be evaluated numerically by employing the data in Table 1. Since the exact values for the points in the plane  $x = 0$  are already known from equation (18), it is possible to check the values in the table by comparison of the two values for  $v_x$ .

The analysis leading to the numerical calculation is based on the fact that the velocity component,  $v_x$ , at a point, P, in the plane  $x = 0$ , due to the infinite cylinder of vortex rings, may be represented by the expression

$$v_{x_p} = \int_0^{4.2} \frac{d\Gamma}{dx} v'_{x_p} dx + \int_{4.2}^{\infty} \frac{d\Gamma}{dx} v'_{x_p} dx.\tag{20}$$

In the above expression,  $\frac{d\Gamma}{dx} v'_{x_p} dx$  is the contribution to  $v_x$  at point,



P, due to a vortex ring of infinitesimal strength,  $\frac{d\Gamma}{dx}$ , at distance  $x$  from the origin of the coordinate system.  $\frac{d\Gamma}{dx}$  is again taken to be  $-1$ , so that the quantity  $\frac{d\Gamma}{dx} v'_{xp}$  is simply the value of the velocity component which a circular vortex of strength  $-1$  would induce at the point, P. This value can be found from Table 1. The first integral in equation (19) may therefore be evaluated by numerical methods since the values of the integrand at a number of  $x$ -values are known. It may be noted that the choice of the arguments in the table is such that Simpson's rule can readily be used in the above-mentioned numerical integration.

The values of the integrand of the second integral are small, but the value of the integral is nevertheless not negligible. The author was unable to express the second integral in closed form. However, an integrable form may be obtained for  $v'_{xp}$  by using an approximate expression for the solid angle which determines  $v'_{xp}$ . Figure 2 shows the geometry used in the derivation of the expression for this approximate value of the solid angle. From the definition on page 5 for the solid angle, it follows that for a point, P, as in the figure, the solid angle is three times that volume of the unit sphere around P which is cut out by the cone determined by P and the vortex ring. The solid angle approximation consists of replacing this common volume of cone and sphere by the volume cut off the cone by a plane parallel to the vortex ring, and intersecting the line connecting P and the center of the ring, at unit distance from P measured along this axis. It can be easily seen that the error involved by using this new volume will be small if the solid angle is small. The approximate expression for  $\omega$  is thus

$$\omega(x, y) \hat{=} \frac{\pi x R^2}{\sqrt{(x^2 + y^2)^3}} \quad (20)$$

which yields

$$\phi(x, y) \hat{=} \frac{\Gamma}{4} \frac{x R^2}{\sqrt{(x^2 + y^2)^3}}. \quad (21)$$

The second integral occurring in equation (19) can be written in the slightly different form,

$$\int_{4.2}^{\infty} v'_{xp} dx = \frac{\partial}{\partial h} \left[ \int_{4.2}^{\infty} \phi(x - h, k) dx \right]_{h=0} \quad (22)$$

The value of  $\phi$  as given by equation (21) when substituted in equation (22) leads to

$$\frac{d\Gamma}{dx} \int_{4.2}^{\infty} v'_{xp} dx = \frac{\partial}{\partial h} \left[ \int_{4.2}^{\infty} \frac{1}{4} \frac{d\Gamma}{dx} \frac{(x - h) dx}{\sqrt{(x - h)^2 + y^2}^3} \right]_{h=0} \quad (23)$$

since  $(x - h)$  is the distance from a ring at distance  $x$  from the reference plane to a point,  $P$ , with distance  $h$  from the reference plane. In analogy with the step from equation (12) to equation (13), equation (22) may be transformed into

$$\frac{d\Gamma}{dx} \int_{4.2}^{\infty} v'_{xp} dx = \frac{\partial}{\partial h} \left[ \int_{4.2+h}^{\infty} \frac{x dx}{\sqrt{(x^2 + y^2)^3}} \right]_{h=0} \quad (24)$$

which reduces to

$$\frac{d\Gamma}{dx} v'_{xp} dx = -\frac{1}{4} \frac{d\Gamma}{dx} \frac{4.2}{\sqrt{(4.2^2 + k^2)^3}}. \quad (25)$$

Equation (25) gives, for  $\frac{d\Gamma}{dx} = -1$ , the approximate value of the velocity induced by that part of the cylinder of vortex rings between  $x = 4.2$  and  $x = \infty$ . The sum of the two integrals in equation (19) will be an approximation to the value of the velocity component induced by the complete vortex ring cylinder. The comparison of this value for  $v_x$  with the exact value given by equation (12) will give some indication of the accuracy of the numerical method. It should be noted, however, that errors introduced by the use of Simpson's rule for the evaluation of the first integral in equation (19) may be partially canceled by the errors due to the approximation for the second integral in equation (19).

It may be remarked here that the use of  $x = 4.2$  as the upper limit in the integral was connected with the fact that this value was the limiting value of the argument in Table 1. The choice of this limiting value was based on the requirement that the errors due to the approximation for the second integral in equation (19) should be sufficiently small. For points on the  $x$ -axis, this integral could be integrated exactly by use of equation (11). Comparison of the exact and approximate values showed that, for the limit value  $x = 4.2$ , the error in the approximation for the second integral amounted to less than 4.5 per cent of the exact value. Since the value of the second integral is small in

comparison with the value of the sum of the integrals, this error was believed to be acceptable. A sample calculation of the error in the sum of the two integrals is given in Appendix III.

## CHAPTER III

## APPLICATIONS

The procedure to obtain the value of the normal component of the induced velocity at an arbitrary point due to an elliptic cylinder of vortex rings will be described with the geometry as given in figure 3.

The normal component of the induced velocity at the point, P, is the integral of the normal components induced by the vortex rings of infinitesimal strength  $\frac{d\Gamma}{dx} dx$  in the wake. The approximate value of this integral,  $\Delta_1 v_{x_p}$ , is obtained by calculating the contribution of the wake for a finite distance from the rotor with Simpson's rule. This is possible since the value of the integrand, i.e., the value of the induced component due to a specific ring, can be found from Table 1. The limited coverage of the tables determines the distance from the rotor to which the numerical integration can be carried. This distance, however, will in most cases be sufficiently large so that the influence,  $\Delta_2 v_{x_p}$ , of the rest of the wake cylinder on the induced velocity at the point, P, will be small. Because of this fact, an approximate expression for the contribution of the remaining part of the wake can be used without introducing an excessive error.

Let the orthogonal coordinate system be chosen such that the origin is at the center of the rotor disk and the rotor disk lies in the plane  $x = 0$ . Let the axis of the skewed cylinder be in the plane  $z = 0$ . The geometry is as in figure 3.



Consider the arbitrary point,  $P(h, k, \ell)$ , and let the center of a specific ring,  $i$ , be given by  $(x_i, y_i)$ . It can be seen that the value of the normal component of the velocity at  $P$  induced by ring  $i$  may be found from the tables, since the normal distance of point  $P$  to the plane of the ring is equal to  $(x - h)$  and the distance from  $P$  to the axis of the ring is given by the expression  $\sqrt{(k - y_i)^2 + \ell^2}$ . If the wake angle is denoted by  $\mathcal{X}$ , then  $y_i = x_i \tan \mathcal{X}$  and  $\sqrt{(k - y_i)^2 + \ell^2}$  becomes  $\sqrt{(k - x \tan \mathcal{X})^2 + \ell^2}$ . In the general case the desired value of the velocity component may be obtained from the tables by a double interpolation.

The integration may be performed by plotting the values of the normal velocity component, induced at the point,  $P$ , by an arbitrary set of rings, against the  $(x_i - h)$ -values of their centers. The integral can in this way be evaluated by use of a planimeter. To save labor the integration may be executed with Simpson's rule. In order to use this rule, it is necessary to choose the set of rings such that the increments in the  $(x_i - h)$  values occur in even sets. The  $x$ -arguments in the table were arranged such that their increments were suitable for application of Simpson's rule. Consequently, the  $x_i$  values should be chosen so that the quantity  $(x_i - h)$  has values that occur as  $x$ -arguments in Table 1. This automatically assures the right increments for use of Simpson's rule and saves one interpolation. Both interpolations may be avoided in the special case of points with  $\ell = 0$ , when the effect of the variation of wake angle is desired. In that case, the wake angles may be chosen such that the values of  $(k - x_i \tan \mathcal{X})$  occur as  $y$ -arguments in Table 1.

The contribution to the induced velocity component due to the remaining part of the wake may be evaluated in the following way. Analogous to

the analysis leading to equation (20), this contribution may be obtained by using an approximate expression for the solid angle. The solid angle subtended at a point,  $P(h, k, \ell)$ , by a ring with radius  $R$  in a plane at distance  $x$  from the reference plane may be taken as

$$\omega_p \hat{=} \frac{\pi R^2 (x - h)}{\sqrt{(x - h)^2 + (k - x \tan \chi)^2 + \ell^2}^3}. \quad (26)$$

If the vortex strength of the ring is assumed to be  $\frac{d\Gamma}{dx} dx$ , then equation (26) leads to

$$d\phi_p = \frac{R^2}{4} \frac{d\Gamma}{dx} \frac{(x - h) dx}{\sqrt{(x - h)^2 + (k - x \tan \chi)^2 + \ell^2}^3}. \quad (27)$$

However, since

$$dv_x = \frac{\partial (d\phi_p)}{\partial h}, \quad (28)$$

equation (28) yields

$$dv_x = \frac{R^2}{4} \frac{d\Gamma}{dx} \left[ \frac{3(x - h)^2 dx}{\sqrt{(x - h)^2 + (k - x \tan \chi)^2 + \ell^2}^5} - \frac{dx}{\sqrt{(x - h)^2 + (k - x \tan \chi)^2 + \ell^2}^3} \right] \quad (29)$$

If the distance from the reference plane to the plane where the remaining part of the wake starts be denoted by  $A$ , the contribution of this

part of the wake will be given by

$$\Delta_2^v x_p = \frac{R^2}{4} \frac{d\Gamma}{dx} \int_A^\infty \frac{3(x-h)^2 dx}{\sqrt{(x-h)^2 + (k-x \tan \chi)^2 + \ell^2}^5} \quad (30)$$

$$- \frac{R^2}{4} \frac{d\Gamma}{dx} \int_A^\infty \frac{dx}{\sqrt{(x-h)^2 + (k-x \tan \chi)^2 + \ell^2}^3}.$$

By use of formulae listed in Peirce (8), equation (30) reduces, after integration and substitution of limits, to

$$\Delta_2^v x_p = \frac{R^2}{2} \frac{d\Gamma}{dx} \left[ \left( \frac{2\sqrt{c}}{q} - \frac{2cA+b}{q\sqrt{X}} \right) \left\{ 1 - \frac{4ac+b^2}{cq} - 2f \left( h^2 + \frac{hb}{c} \right) \right\} \right. \quad (31)$$

$$\left. + \frac{h}{cX\sqrt{X}} + \frac{2cA+b}{qX\sqrt{X}} \left( h^2 + \frac{hb}{c} \right) + \frac{(b^2 - 2ac)A + ab}{cqX\sqrt{X}} \right]$$

where

$$a = h^2 + k^2 + \ell^2,$$

$$b = -2(h + k \tan \chi),$$

$$c = 1 + \tan^2 \chi,$$

$$q = 4ac - b^2,$$

$$f = \frac{4c}{q}$$

$$X = a + bA + cA^2.$$

Equation (31) gives the value of  $v_{x_p}$  for the general case. For practical purposes labor may be saved by simplifying equation (31). This may be achieved by a translation of the coordinate system such that point P lies in the plane  $x = 0$  and the origin of the coordinate system lies on



the axis of the wake; in this case the contribution to the velocity at P will be expressed by equation (31), if the new coordinates for the point, P, are used and a new limit of integration A' is used. From the geometry in figure 3 it can be seen that the new coordinates of P are related to the old ones by the following expressions:

$$\begin{aligned}h' &= 0, \\k' &= k - h \tan \chi, \\l' &= l, \\A' &= A - h,\end{aligned}$$

As  $h' = 0$ , equation (31) reduces to

$$\Delta_2 v_{x_p} = \frac{R^2}{2} \frac{d\Gamma}{dx} \left[ \left( \frac{2\sqrt{c}}{q'} - \frac{2cA' + b'}{q\sqrt{x'}} \right) \left\{ 1 - \frac{4a'c + b'^2}{cq} \right\} + \frac{(b'^2 - 2a'c)A' + a'b'}{cq'x'\sqrt{x'}} \right] \quad (32)$$

where

$$\begin{aligned}a' &= k'^2 + l'^2, \\b' &= -2k' \tan \chi, \\q' &= 4a'c - b'^2, \\x' &= a' + b'A' + cA'^2.\end{aligned}$$

From equation (32) it may be seen that the value of  $v_{x_p}$  becomes indeterminate for the special point, P (0,0,0). It is possible to get an expression for  $v_{x_p}$  at this point by substituting the coordinates of the

point in equation (30). The result is the simplified equation

$$\Delta_2 v_{x_p} = -\frac{R^2}{4} \frac{d\Gamma}{dx} \int_A^\infty \frac{3\cos^5 \chi - \cos^3 \chi}{x^3} dx. \quad (33)$$

Integration of equation (33) and substitution of the limits leads to

$$\Delta_2 v_{x_p} = -\frac{R^2}{4} \frac{d\Gamma}{dx} \cos^3 \chi (3\cos^2 \chi - 1) \frac{1}{2A^2}. \quad (34)$$

In Appendix IV sample calculations are shown for the evaluation of the normal induced velocity component at the points  $P_1 (0, 0.1, 0)$ ,  $P_2 (0, -0.1, 0)$  and  $P_3 (0, 0, 0)$  in the plane of a rotor with radius  $R = 1$  for the case where the wake angle  $\chi = \arctan 0.5$ . Coleman, Feingold and Stempin (9) proved that the exact value of the normal component at the center of the rotor due to the cylindrical type of wake is

$$v_{x_{\text{center}}} = \frac{1}{2} \cos \chi \frac{d\Gamma}{dx}. \quad (35)$$

The rate of change of the normal velocity component along the Z-axis at the center of the rotor is

$$\left( \frac{\partial v_x}{\partial z} \right)_{\text{center}} = \frac{1}{2} \frac{d\Gamma}{dx} \cos \chi \tan \frac{\chi}{2} \quad (36)$$

Comparison of the numerical results for the quantities at the center of the rotor, which were obtained from the calculated values of  $v_{x_p}$  at the three points, with the exact values as found from equations (35) and (36) showed close agreement.

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## APPENDIX I

Table 1. Table of Induced Velocities.

x	y =	Induced velocities					
		0.0	0.1	0.2	0.3	0.4	0.5
0.0		0.5000	0.5038	0.5156	0.5369	0.5707	0.6228
0.1		0.4926	0.4961	0.5070	0.5264	0.5569	0.6025
0.2		0.4714	0.4742	0.4827	0.4974	0.5193	0.5494
0.4		0.4002	0.4010	0.4032	0.4064	0.4093	0.4098
0.6		0.3153	0.3147	0.3128	0.3089	0.3022	0.2911
0.8		0.2381	0.2370	0.2338	0.2281	0.2196	0.2076
1.0		0.1768	0.1758	0.1728	0.1677	0.1604	0.1508
1.3		0.1133	0.1127	0.1106	0.1073	0.0927	0.0969
1.6		0.0744	0.0740	0.0728	0.0708	0.0681	0.0647
2.1		0.0397	0.0396	0.0391	0.0382	0.0371	0.0357
2.6		0.0231	0.0231	0.0228	0.0225	0.0220	0.0213
3.4		0.0112	0.0112	0.0111	0.0110	0.0109	0.0106
4.2		0.0062	0.0062	0.0062	0.0061	0.0061	0.0060
x	y =	0.6	0.7	0.8	0.9	1.0	1.1
0.0		0.7053	0.08461	1.1293	1.9630	$\infty$	-1.2627
0.1		0.6711	0.7768	0.9397	1.0938	0.2687	-0.5298
0.2		0.5881	0.6304	0.6496	0.5501	0.2126	-0.1094
0.4		0.4034	0.3825	0.3365	0.2568	0.1550	0.0571
0.6		0.2735	0.2476	0.2121	0.1682	0.1201	0.0747
0.8		0.1919	0.1721	0.1486	0.1225	0.0956	0.0724
1.0		0.1391	0.1253	0.1099	0.0934	0.0768	0.0579
1.3		0.0900	0.0822	0.0738	0.0649	0.0560	0.0473
1.6		0.0608	0.0563	0.0516	0.0465	0.0415	0.0364
2.1		0.0341	0.0322	0.0302	0.0281	0.0259	0.0236
2.6		0.0206	0.0197	0.0188	0.0178	0.0168	0.0156
3.4		0.0104	0.0098	0.0095	0.0091	0.0087	0.0083
4.2		0.0059	0.0058	0.0056	0.0055	0.0054	0.0052

(CONTINUED)



Table 1. Table of Induced Velocities. (Continued)

		Induced Velocities					
<u>x</u>	<u>y =</u>	<u>1.2</u>	<u>1.3</u>	<u>1.4</u>	<u>1.5</u>	<u>1.6</u>	<u>1.7</u>
0.0		-0.5324	-0.3062	-0.2010	-0.1424	-0.1060	-0.0817
0.1		-0.3951	-0.2633	-0.1833	-0.1336	-0.1011	-0.0788
0.2		-0.1941	-0.1756	-0.1411	-0.1112	-0.0882	-0.0709
0.4		-0.0096	-0.0438	-0.0559	-0.0567	-0.0527	-0.0471
0.6		0.0371	0.0099	-0.0076	-0.0175	-0.0218	-0.0241
0.8		0.0470	0.0284	0.0041	0.0038	-0.0031	-0.0076
1.0		0.0460	0.0332	0.0226	0.0141	0.0073	0.0026
1.3		0.0391	0.0316	0.0249	0.0190	0.0141	0.0100
1.6		0.0315	0.0269	0.0226	0.0187	0.0153	0.0122
2.1		0.0209	0.0192	0.0171	0.0151	0.0132	0.0115
2.6		0.0146	0.0135	0.0124	0.0113	0.0103	0.0093
3.4		0.0083	0.0079	0.0075	0.0070	0.0066	0.0062
4.2		0.0050	0.0048	0.0047	0.0045	0.0043	0.0041

		Induced Velocities						
<u>x</u>	<u>y =</u>	<u>1.8</u>	<u>1.9</u>	<u>2.0</u>	<u>2.6</u>	<u>3.2</u>	<u>4.0</u>	<u>5.0</u>
0.0		-0.0647	-0.0524	-0.0431	-0.0170	-0.0086	-0.0042	-0.0021
0.1		-0.0629	-0.0511	-0.0423	-0.0169	-0.0085	-0.0042	-0.0021
0.2		-0.0577	-0.0477	-0.0398	-0.0164	-0.0084	-0.0041	-0.0021
0.4		-0.0414	-0.0362	-0.0315	-0.0147	-0.0079	-0.0040	-0.0020
0.6		-0.0241	-0.0230	-0.0215	-0.0123	-0.0071	-0.0037	-0.0020
0.8		-0.0103	-0.0117	-0.0122	-0.0095	-0.0061	-0.0034	-0.0018
1.0		-0.0009	-0.0034	-0.0050	-0.0067	-0.0050	-0.0030	-0.0017
1.3		0.0067	0.0040	0.0020	-0.0032	-0.0034	-0.0024	-0.0015
1.6		0.0096	0.0074	0.0060	-0.0005	-0.0019	-0.0018	-0.0013
2.1		0.0099	0.0085	0.0072	0.0020	-0.0001	-0.0009	-0.0008
2.6		0.0084	0.0075	0.0067	0.0029	0.0010	-0.0001	-0.0005
3.4		0.0058	0.0054	0.0050	0.0030	0.0016	0.0005	0.0000
4.2		0.0039	0.0037	0.0035	0.0024	0.0019	0.0009	0.0002

## APPENDIX II

The differentiation of the expression for  $\psi$  in equation (3) will be accomplished in parts. According to equation (1),  $v_x = -\frac{1}{y} \frac{\partial \psi}{\partial y}$ .

Using the expression for  $\psi$ ,  $v_x$  may be written

$$v_x = \frac{\Gamma}{2\pi y} \left[ \left\{ K(\lambda) - E(\lambda) \right\} \frac{\partial}{\partial y} (r_1 + r_2) + (r_1 + r_2) \frac{d}{d\lambda} \left\{ K(\lambda) - E(\lambda) \right\} \frac{\partial \lambda}{\partial y} \right]. \quad (37)$$

The operations in equation (37) will be divided by introducing the quantities

$$\begin{aligned} A &= K(\lambda) - E(\lambda), \\ B &= \frac{\partial}{\partial y} (r_1 + r_2), \\ C &= r_1 + r_2, \\ D &= \frac{d}{d\lambda} \left\{ K(\lambda) - E(\lambda) \right\}, \\ E &= \frac{\partial \lambda}{\partial y}. \end{aligned}$$

The quantities  $r_1$  and  $r_2$  are given by the expressions  $\sqrt{x^2 + (y - R)^2}$  and  $\sqrt{x^2 + (y + R)^2}$  respectively. Hence, the quantity B may be shown to be

$$B = \frac{y - R}{\sqrt{x^2 + (y - R)^2}} + \frac{y + R}{\sqrt{x^2 + (y + R)^2}}. \quad (38)$$

C was shown previously to be given by

$$C = \sqrt{x^2 + (y - R)^2} + \sqrt{x^2 + (y + R)^2}. \quad (39)$$

The differentiation of  $\lambda$  is straightforward and results in

$$E = \frac{\partial \lambda}{\partial y} = \frac{1}{R} - \frac{(x^2 + y^2 + R^2) - \sqrt{x^2 + (y - R)^2} \sqrt{x^2 + (y + R)^2}}{2Ry^2} \quad (40)$$

$$- \frac{(y + R) \{x^2 + (y - R)^2\} + (y - R) \{x^2 + (y + R)^2\}}{2Ry \sqrt{x^2 + (y - R)^2} \sqrt{x^2 + (y + R)^2}}.$$

The differentiation of the elliptic integrals is as follows.

$$\frac{dK(\lambda)}{d\lambda} = \frac{d}{d\lambda} \int_0^1 \frac{dt}{\sqrt{1-t^2} \sqrt{1-\lambda^2 t^2}}. \quad (41)$$

The interchange of differentiation and integration is allowed and therefore

$$\frac{dK(\lambda)}{d\lambda} = \int_0^1 \frac{\lambda t^2 dx}{\sqrt{1-t^2} \sqrt{(1-\lambda^2 t^2)^3}}. \quad (42)$$

Equation (42) may also be written

$$\begin{aligned} \frac{dK(\lambda)}{d\lambda} = & -\frac{1}{\lambda} \int_0^1 \frac{dt}{\sqrt{1-t^2} \sqrt{(1-\lambda^2 t^2)}} \\ & + \frac{1}{\lambda} \int_0^1 \frac{dt}{\sqrt{1-t^2} \sqrt{(1-\lambda^2 t^2)^3}}. \end{aligned} \quad (43)$$

Magnus (10) gives an expression for the second integral of equation (43):

$$\int_0^y \frac{dt}{\sqrt{1-t^2} \sqrt{(1-\lambda^2 t^2)^3}} = \frac{E}{\lambda'^2} - \frac{\lambda^2}{\lambda'^2} \frac{y \sqrt{1-y^2}}{\sqrt{1-\lambda^2 y^2}} \quad (44)$$

where

$$\lambda' = \sqrt{1-\lambda^2}.$$

In the case here under consideration  $y = 1$  and equation (44) reduces to

$$\int_0^1 \frac{dt}{\sqrt{1-t^2} \sqrt{(1-\lambda^2 t^2)^3}} = \frac{E}{1-\lambda^2} \quad (45)$$

without complications, except for the case  $\lambda = 1$ . Since this value of  $\lambda$  will never be reached in the calculations in this paper, this complication can be disregarded. The resulting expression for  $\frac{dK(\lambda)}{d\lambda}$  may be seen to be

$$\frac{dK(\lambda)}{d\lambda} = -\frac{1}{\lambda} K(\lambda) + \frac{E(\lambda)}{(1-\lambda^2)}. \quad (46)$$

The derivative of the elliptic integral,  $E(\lambda)$ , with respect to  $\lambda$  can be represented by

$$\frac{dE(\lambda)}{d\lambda} = \frac{d}{d\lambda} \int_0^1 \frac{\sqrt{1-\lambda^2 t^2}}{\sqrt{1-t^2}} dt. \quad (47)$$

Differentiation and integration may again be interchanged to obtain

$$\frac{dE(\lambda)}{d\lambda} = \int_0^1 \frac{-\lambda t^2}{\sqrt{1-t^2} \sqrt{1-\lambda^2 t^2}} dt. \quad (48)$$



Equation (48) may be transformed into

$$\frac{dE(\lambda)}{d\lambda} = \frac{1}{\lambda} \int_0^1 \frac{\sqrt{1 - \lambda^2 t^2}}{\sqrt{1 - t^2}} dt - \frac{1}{\lambda} \int_0^1 \frac{dt}{\sqrt{1 - t^2} \sqrt{1 - \lambda^2 t^2}}. \quad (49)$$

Equation (49) reduces to

$$\frac{dE(\lambda)}{d\lambda} = \frac{1}{\lambda} \left\{ E(\lambda) - K(\lambda) \right\}. \quad (50)$$

One obtains by use of equations (46) and (50) the final result for

$\frac{d}{d\lambda} (K - E)$  which is

$$\frac{d}{d\lambda} \left\{ K(\lambda) - E(\lambda) \right\} = D = \frac{\lambda E(\lambda)}{1 - \lambda^2}. \quad (51)$$

The necessary additional differentiations for  $v_y$  do not introduce any difficulties and are therefore omitted.

## APPENDIX III

The approximate value of the normal component of velocity induced by a straight cylinder of vortex rings will be computed here for the point P (0,0.7). The first integral of equation (19) is evaluated in tabular form. The values of  $v'_{x_p}$  as taken from Table 1 are:

(1) x	(2) y	(3) $v'_{x_p}$	(4) Simpson's Coefficient times six
0.0	0.7	0.8461	0.2
0.1	0.7	0.7768	0.8
0.2	0.7	0.6304	0.6
0.4	0.7	0.3825	1.6
0.6	0.7	0.2476	0.8
0.8	0.7	0.1721	1.6
1.0	0.7	0.1253	1.0
1.3	0.7	0.0822	2.4
1.6	0.7	0.0563	1.6
2.1	0.7	0.0322	4.0
2.6	0.7	0.0197	2.6
3.4	0.7	0.0101	6.4
4.2	0.7	0.0058	1.6

The sum of the products of columns (3) and (4) is equal to six times the first integral of equation (19). The value is 2.92094 from which the value of the first integral is found to be 0.4868. The value of the second integral is computed from equation (25). This results in a value of 0.0136. The sum of the two integrals, i.e., the value of  $v_x$  at P (0,0.7), is therefore  $0.4868 + 0.0136 = 0.5004$ . Equation (18) shows that the exact value of  $v_x$  at this point is 0.5000. The approximate value checks within a high degree of accuracy.

## APPENDIX IV

The following calculations are for the normal induced velocity component at the points  $P_1 (0,0.1,0)$ ,  $P_2 (0,-0.1,0)$  and  $P_3 (0,0,0)$  for the case of a rotor with unit radius, which has a wake consisting of a skewed elliptic cylinder. The strength of the wake per unit length is assumed to be  $\frac{d\Gamma}{dx} = -1$  and the wake angle  $\chi = \arctan 0.5$ . The contribution to the normal velocity component due to the finite length of the wake is calculated for each point in tabular form. The increment due to the remaining section of the wake is calculated by using the appropriate equations.

$P_1 (0,0.1,0)$

(1) $x_1$	(2) $x_1 - h$	(3) $\sqrt{(k - x \tan \chi)^2 + \ell^2}$	(4) $v_{x_p}$	(5) Simpson's Coefficient times six
0.0	0.0	0.1	0.4961	0.2
0.1	0.1	0.05	0.4900	0.8
0.2	0.2	0.0	0.4714	0.6
0.4	0.4	-0.1	0.4010	1.6
0.6	0.6	-0.2	0.3128	0.8
0.8	0.8	-0.3	0.2281	1.6
1.0	1.0	-0.4	0.1604	1.0
1.3	1.3	-0.55	0.0930	2.4
1.6	1.6	-0.7	0.0563	1.6
2.1	2.1	-0.95	0.0270	4.0
2.6	2.6	-1.2	0.0146	2.6
3.4	3.4	-1.6	0.0066	6.4
4.2	4.2	-2.0	0.0035	1.6

The total sum of the product of column (4) and column (5) is six times the value of  $v_{x_p}$  and has the value 2.69834. Therefore,  $\Delta_1 v_{x_p} =$

0.4497.  $\Delta_2 v_{xp}$  may be calculated from equation (32). The quantities that occur in that equation were calculated and are

$$\begin{aligned} a' &= 0.01 \\ b' &= -0.1 \\ c &= 1.25 \\ q &= 0.04 \\ X &= 21.64 \end{aligned}$$

Substituting these values in equation (32) results in  $v_{xp1} = 0.0074$ . The total value of  $v_{xp1}$  is  $v_{xp1} = 0.4571$ .

$$\underline{P_2 (0, -0.1, 0)}$$

(1)	(2)	(3)	(4)	(5)
$x_i$	$x_i - h$	$\sqrt{(k - x \tan X)^2 + \ell^2}$	$v_{xp}$	Simpson's Coefficient times six
0.0	0.0	-0.1	0.4961	0.2
0.1	0.1	-0.15	0.5010	0.8
0.2	0.2	-0.2	0.4827	0.6
0.4	0.4	-0.3	0.4064	1.6
0.6	0.6	-0.4	0.3022	0.8
0.8	0.8	-0.5	0.2076	1.6
1.0	1.0	-0.6	0.1391	1.0
1.3	1.3	-0.75	0.0781	2.4
1.6	1.6	-0.9	0.0465	1.6
2.1	2.1	-1.15	0.0223	4.0
2.6	2.6	-1.4	0.0124	2.6
3.4	3.4	-1.8	0.0058	6.4
4.2	4.2	-2.2	0.0031	1.6

The sum of the products of columns (4) and (5) is 2.578626 or  $v_{xp2} = 0.4297$ . The quantities occurring in equation (32) are found to be

$$\begin{aligned} a' &= 0.01 \\ b' &= 0.1 \\ c &= 1.25 \\ q &= 0.04 \\ X &= 22.48 \end{aligned}$$

Upon substitution of these values in equation (32), one obtains  $\Delta_2 v_{xp2} =$

0.0068, which means that the total value  $v_{xp2} = 0.4365$ .

$$P_3 (0,0,0)$$

(1)	(2)	(3)	(4)	(5)
$x_i$	$x_i - h$	$\sqrt{(k - x \tan \alpha)^2 + l^2}$	$v_{xp}$	Simpson's Coefficient times six
0.0	0.0	0.0	0.5000	0.2
0.1	0.1	-0.05	0.4945	0.8
0.2	0.2	-0.1	0.4742	1.6
0.4	0.4	-0.2	0.4032	1.6
0.6	0.6	-0.3	0.3089	0.8
0.8	0.8	-0.4	0.2196	1.6
1.0	1.0	-0.5	0.1508	1.0
1.3	1.3	-0.65	0.0865	2.4
1.6	1.6	-0.8	0.0516	4.0
2.1	2.1	-1.05	0.0248	2.6
2.6	2.6	-1.3	0.0135	6.4
3.4	3.4	-1.7	0.0062	1.6
4.2	4.2	-2.1	0.0033	1.6

The sum of products of columns (4) and (5) is in this case equal to 2.64394, which leads to  $\Delta_1 v_{xp3} = 0.4407$ . For this point equation (34) was utilized to evaluate  $\Delta_2 v_{xp3}$ ; the result is  $\Delta_2 v_{xp3} = 0.0071$ . Therefore  $v_{xp3} = 0.4478$ .

An approximation for the quantity  $\left(\frac{\partial v_x}{\partial z}\right)_{P_3}$  may be found by using the finite difference between  $v_{xp1}$  and  $v_{xp2}$  divided by the distance of the points  $P_1$  and  $P_2$ . This results in

$$\left(\frac{\partial v_x}{\partial z}\right)_{P_3} \hat{=} 0.103$$

The exact values for the quantities  $v_{xp3}$  and  $\left(\frac{\partial v_x}{\partial z}\right)_{P_3}$  may be found from from equations (35) and (36). The results are

$$\left(\frac{\partial v_x}{\partial z}\right)_{P_3} = 0.106$$

$$v_{x_{p_3}} = 0.4472.$$

The comparison between the two values for  $v_{x_{p_3}}$  is seen to be very close. The approximate value of  $\left(\frac{\partial v_x}{\partial z}\right)_{p_3}$  is slightly less accurate than the value for the induced velocity. However, this might be expected since the procedure for evaluating  $\left(\frac{\partial v_x}{\partial z}\right)_{p_3}$  introduces additional error.

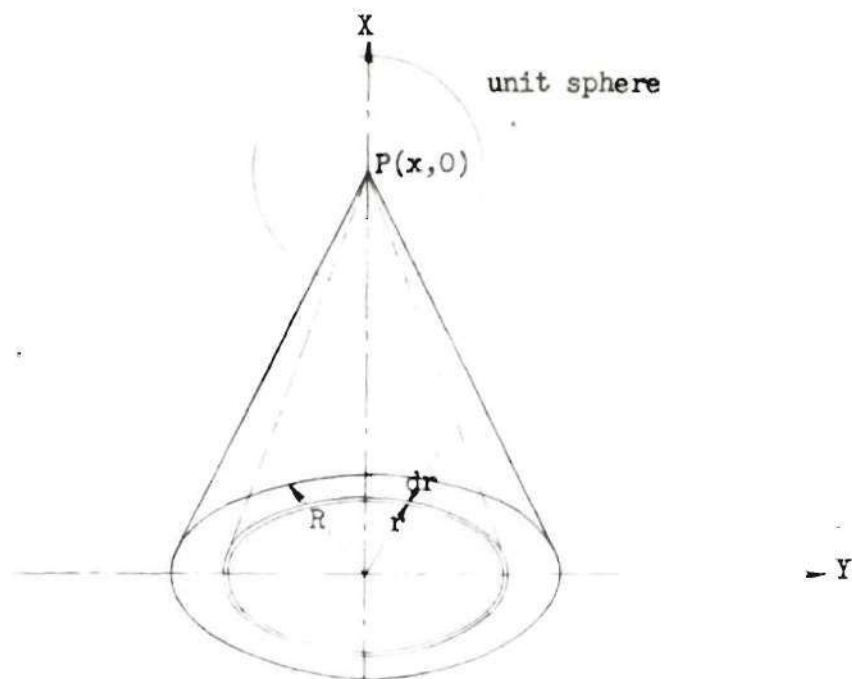


Figure 1  
Geometry for the Derivation of the Exact Expression  
for the Solid Angle.

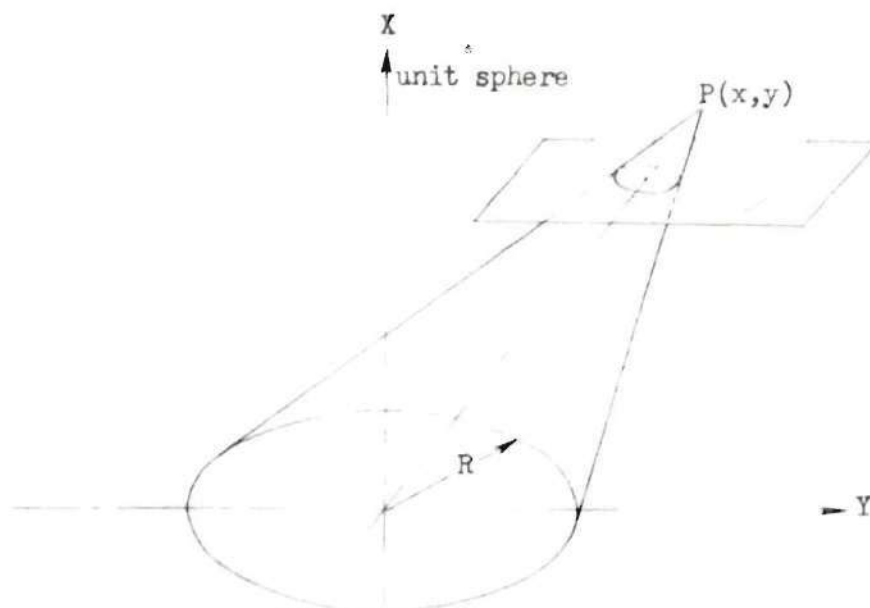


Figure 2  
Geometry for the Derivation of the Approximate Expression  
for the Solid Angle.



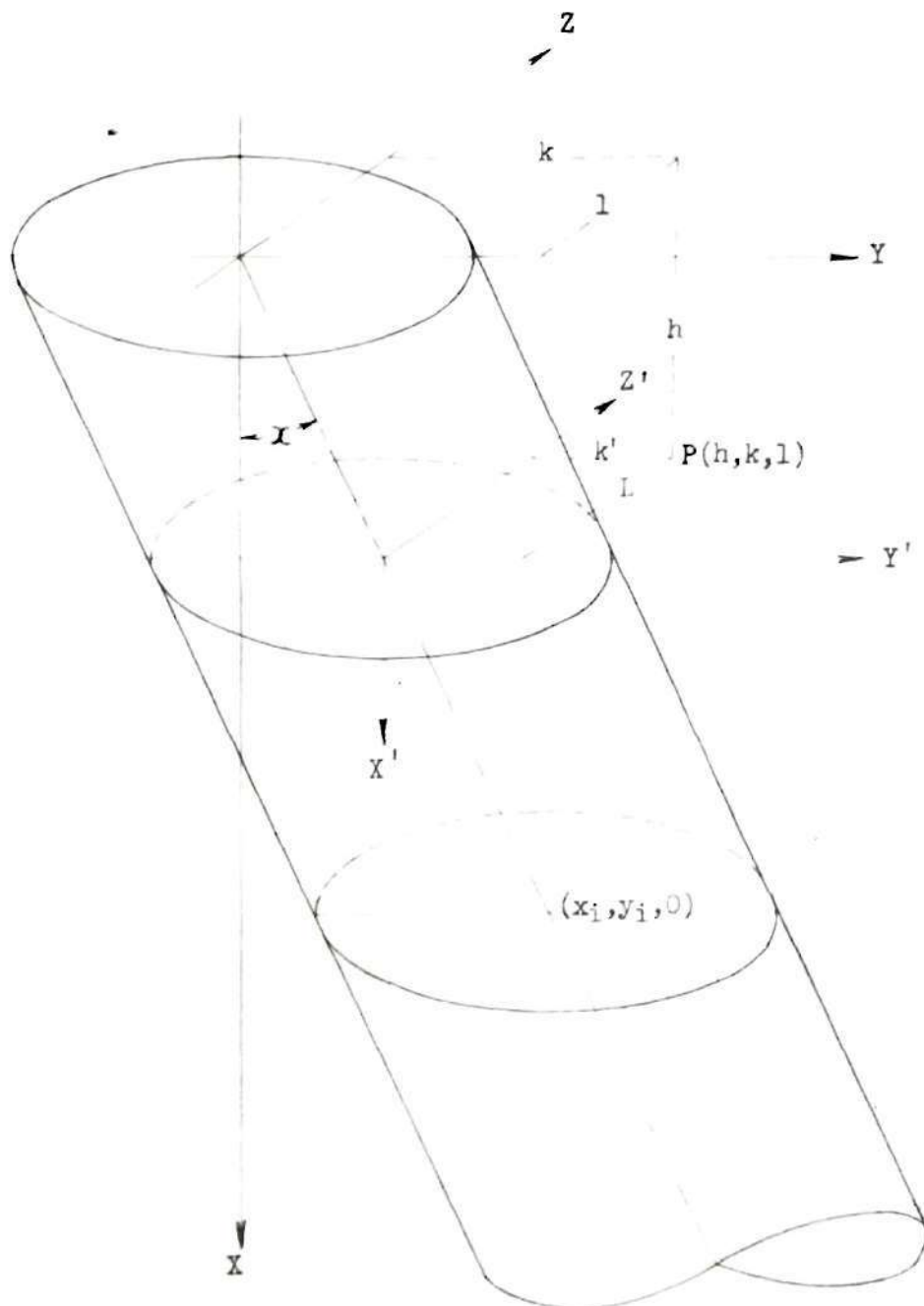


Figure 3  
Geometry of Rotor and Wake.